

International Series on Actuarial Science

Actuarial Mathematics for Life Contingent Risks

David C. M. Dickson, Mary R. Hardy
and Howard R. Waters

CAMBRIDGE



CAMBRIDGE

www.cambridge.org/9780521118255

This page intentionally left blank

Actuarial Mathematics for Life Contingent Risks

How can actuaries best equip themselves for the products and risk structures of the future? In this new textbook, three leaders in actuarial science give a modern perspective on life contingencies.

The book begins traditionally, covering actuarial models and theory, and emphasizing practical applications using computational techniques. The authors then develop a more contemporary outlook, introducing multiple state models, emerging cash flows and embedded options. Using spreadsheet-style software, the book presents large-scale, realistic examples. Over 150 exercises and solutions teach skills in simulation and projection through computational practice.

Balancing rigour with intuition, and emphasizing applications, this textbook is ideal not only for university courses, but also for individuals preparing for professional actuarial examinations and qualified actuaries wishing to renew and update their skills.

International Series on Actuarial Science

Christopher Daykin, Independent Consultant and Actuary
Angus Macdonald, Heriot-Watt University

The *International Series on Actuarial Science*, published by Cambridge University Press in conjunction with the Institute of Actuaries and the Faculty of Actuaries, contains textbooks for students taking courses in or related to actuarial science, as well as more advanced works designed for continuing professional development or for describing and synthesizing research. The series is a vehicle for publishing books that reflect changes and developments in the curriculum, that encourage the introduction of courses on actuarial science in universities, and that show how actuarial science can be used in all areas where there is long-term financial risk.

ACTUARIAL MATHEMATICS FOR LIFE CONTINGENT RISKS

DAVID C. M. DICKSON

University of Melbourne

MARY R. HARDY

University of Waterloo, Ontario

HOWARD R. WATERS

Heriot-Watt University, Edinburgh



CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org

Information on this title: www.cambridge.org/9780521118255

© D. C. M. Dickson, M. R. Hardy and H. R. Waters 2009

This publication is in copyright. Subject to statutory exception and to the provision of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published in print format 2009

ISBN-13 978-0-511-65169-4 eBook (NetLibrary)

ISBN-13 978-0-521-11825-5 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of urls for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

*To
Carolann,
Vivien
and Phelim*

Contents

<i>Preface</i>	<i>page</i>	<i>xiv</i>
1	Introduction to life insurance	1
1.1	Summary	1
1.2	Background	1
1.3	Life insurance and annuity contracts	3
1.3.1	Introduction	3
1.3.2	Traditional insurance contracts	4
1.3.3	Modern insurance contracts	6
1.3.4	Distribution methods	8
1.3.5	Underwriting	8
1.3.6	Premiums	10
1.3.7	Life annuities	11
1.4	Other insurance contracts	12
1.5	Pension benefits	12
1.5.1	Defined benefit and defined contribution pensions	13
1.5.2	Defined benefit pension design	13
1.6	Mutual and proprietary insurers	14
1.7	Typical problems	14
1.8	Notes and further reading	15
1.9	Exercises	15
2	Survival models	17
2.1	Summary	17
2.2	The future lifetime random variable	17
2.3	The force of mortality	21
2.4	Actuarial notation	26
2.5	Mean and standard deviation of T_x	29
2.6	Curtate future lifetime	32
2.6.1	K_x and e_x	32

	2.6.2	The complete and curtate expected future lifetimes, $\overset{\circ}{e}_x$ and e_x	34
	2.7	Notes and further reading	35
	2.8	Exercises	36
3		Life tables and selection	41
	3.1	Summary	41
	3.2	Life tables	41
	3.3	Fractional age assumptions	44
		3.3.1 Uniform distribution of deaths	44
		3.3.2 Constant force of mortality	48
	3.4	National life tables	49
	3.5	Survival models for life insurance policyholders	52
	3.6	Life insurance underwriting	54
	3.7	Select and ultimate survival models	56
	3.8	Notation and formulae for select survival models	58
	3.9	Select life tables	59
	3.10	Notes and further reading	67
	3.11	Exercises	67
4		Insurance benefits	73
	4.1	Summary	73
	4.2	Introduction	73
	4.3	Assumptions	74
	4.4	Valuation of insurance benefits	75
		4.4.1 Whole life insurance: the continuous case, \bar{A}_x	75
		4.4.2 Whole life insurance: the annual case, A_x	78
		4.4.3 Whole life insurance: the 1/ <i>m</i> thly case, $A_x^{(m)}$	79
		4.4.4 Recursions	81
		4.4.5 Term insurance	86
		4.4.6 Pure endowment	88
		4.4.7 Endowment insurance	89
		4.4.8 Deferred insurance benefits	91
	4.5	Relating \bar{A}_x , A_x and $A_x^{(m)}$	93
		4.5.1 Using the uniform distribution of deaths assumption	93
		4.5.2 Using the claims acceleration approach	95
	4.6	Variable insurance benefits	96
	4.7	Functions for select lives	101
	4.8	Notes and further reading	101
	4.9	Exercises	102
5		Annuities	107
	5.1	Summary	107
	5.2	Introduction	107

5.3	Review of annuities-certain	108
5.4	Annual life annuities	108
5.4.1	Whole life annuity-due	109
5.4.2	Term annuity-due	112
5.4.3	Whole life immediate annuity	113
5.4.4	Term immediate annuity	114
5.5	Annuities payable continuously	115
5.5.1	Whole life continuous annuity	115
5.5.2	Term continuous annuity	117
5.6	Annuities payable m times per year	118
5.6.1	Introduction	118
5.6.2	Life annuities payable m times a year	119
5.6.3	Term annuities payable m times a year	120
5.7	Comparison of annuities by payment frequency	121
5.8	Deferred annuities	123
5.9	Guaranteed annuities	125
5.10	Increasing annuities	127
5.10.1	Arithmetically increasing annuities	127
5.10.2	Geometrically increasing annuities	129
5.11	Evaluating annuity functions	130
5.11.1	Recursions	130
5.11.2	Applying the UDD assumption	131
5.11.3	Woolhouse's formula	132
5.12	Numerical illustrations	135
5.13	Functions for select lives	136
5.14	Notes and further reading	137
5.15	Exercises	137
6	Premium calculation	142
6.1	Summary	142
6.2	Preliminaries	142
6.3	Assumptions	143
6.4	The present value of future loss random variable	145
6.5	The equivalence principle	146
6.5.1	Net premiums	146
6.6	Gross premium calculation	150
6.7	Profit	154
6.8	The portfolio percentile premium principle	162
6.9	Extra risks	165
6.9.1	Age rating	165
6.9.2	Constant addition to μ_x	165
6.9.3	Constant multiple of mortality rates	167

6.10	Notes and further reading	169
6.11	Exercises	170
7	Policy values	176
7.1	Summary	176
7.2	Assumptions	176
7.3	Policies with annual cash flows	176
7.3.1	The future loss random variable	176
7.3.2	Policy values for policies with annual cash flows	182
7.3.3	Recursive formulae for policy values	191
7.3.4	Annual profit	196
7.3.5	Asset shares	200
7.4	Policy values for policies with cash flows at discrete intervals other than annually	203
7.4.1	Recursions	204
7.4.2	Valuation between premium dates	205
7.5	Policy values with continuous cash flows	207
7.5.1	Thiele's differential equation	207
7.5.2	Numerical solution of Thiele's differential equation	211
7.6	Policy alterations	213
7.7	Retrospective policy value	219
7.8	Negative policy values	220
7.9	Notes and further reading	220
7.10	Exercises	220
8	Multiple state models	230
8.1	Summary	230
8.2	Examples of multiple state models	230
8.2.1	The alive–dead model	230
8.2.2	Term insurance with increased benefit on accidental death	232
8.2.3	The permanent disability model	232
8.2.4	The disability income insurance model	233
8.2.5	The joint life and last survivor model	234
8.3	Assumptions and notation	235
8.4	Formulae for probabilities	239
8.4.1	Kolmogorov's forward equations	242
8.5	Numerical evaluation of probabilities	243
8.6	Premiums	247
8.7	Policy values and Thiele's differential equation	250
8.7.1	The disability income model	251
8.7.2	Thiele's differential equation – the general case	255

	8.8 Multiple decrement models	256
	8.9 Joint life and last survivor benefits	261
	8.9.1 The model and assumptions	261
	8.9.2 Joint life and last survivor probabilities	262
	8.9.3 Joint life and last survivor annuity and insurance functions	264
	8.9.4 An important special case: independent survival models	270
	8.10 Transitions at specified ages	274
	8.11 Notes and further reading	278
	8.12 Exercises	279
9	Pension mathematics	290
	9.1 Summary	290
	9.2 Introduction	290
	9.3 The salary scale function	291
	9.4 Setting the DC contribution	294
	9.5 The service table	297
	9.6 Valuation of benefits	306
	9.6.1 Final salary plans	306
	9.6.2 Career average earnings plans	312
	9.7 Funding plans	314
	9.8 Notes and further reading	319
	9.9 Exercises	319
10	Interest rate risk	326
	10.1 Summary	326
	10.2 The yield curve	326
	10.3 Valuation of insurances and life annuities	330
	10.3.1 Replicating the cash flows of a traditional non-participating product	332
	10.4 Diversifiable and non-diversifiable risk	334
	10.4.1 Diversifiable mortality risk	335
	10.4.2 Non-diversifiable risk	336
	10.5 Monte Carlo simulation	342
	10.6 Notes and further reading	348
	10.7 Exercises	348
11	Emerging costs for traditional life insurance	353
	11.1 Summary	353
	11.2 Profit testing for traditional life insurance	353
	11.2.1 The net cash flows for a policy	353
	11.2.2 Reserves	355
	11.3 Profit measures	358
	11.4 A further example of a profit test	360

11.5	Notes and further reading	369
11.6	Exercises	369
12	Emerging costs for equity-linked insurance	374
12.1	Summary	374
12.2	Equity-linked insurance	374
12.3	Deterministic profit testing for equity-linked insurance	375
12.4	Stochastic profit testing	384
12.5	Stochastic pricing	388
12.6	Stochastic reserving	390
12.6.1	Reserving for policies with non-diversifiable risk	390
12.6.2	Quantile reserving	391
12.6.3	CTE reserving	393
12.6.4	Comments on reserving	394
12.7	Notes and further reading	395
12.8	Exercises	395
13	Option pricing	401
13.1	Summary	401
13.2	Introduction	401
13.3	The ‘no arbitrage’ assumption	402
13.4	Options	403
13.5	The binomial option pricing model	405
13.5.1	Assumptions	405
13.5.2	Pricing over a single time period	405
13.5.3	Pricing over two time periods	410
13.5.4	Summary of the binomial model option pricing technique	413
13.6	The Black–Scholes–Merton model	414
13.6.1	The model	414
13.6.2	The Black–Scholes–Merton option pricing formula	416
13.7	Notes and further reading	427
13.8	Exercises	428
14	Embedded options	431
14.1	Summary	431
14.2	Introduction	431
14.3	Guaranteed minimum maturity benefit	433
14.3.1	Pricing	433
14.3.2	Reserving	436
14.4	Guaranteed minimum death benefit	438
14.4.1	Pricing	438
14.4.2	Reserving	440

14.5	Pricing methods for embedded options	444
14.6	Risk management	447
14.7	Emerging costs	449
14.8	Notes and further reading	457
14.9	Exercises	458
A	Probability theory	464
A.1	Probability distributions	464
A.1.1	Binomial distribution	464
A.1.2	Uniform distribution	464
A.1.3	Normal distribution	465
A.1.4	Lognormal distribution	466
A.2	The central limit theorem	469
A.3	Functions of a random variable	469
A.3.1	Discrete random variables	470
A.3.2	Continuous random variables	470
A.3.3	Mixed random variables	471
A.4	Conditional expectation and conditional variance	472
A.5	Notes and further reading	473
B	Numerical techniques	474
B.1	Numerical integration	474
B.1.1	The trapezium rule	474
B.1.2	Repeated Simpson's rule	476
B.1.3	Integrals over an infinite interval	477
B.2	Woolhouse's formula	478
B.3	Notes and further reading	479
C	Simulation	480
C.1	The inverse transform method	480
C.2	Simulation from a normal distribution	481
C.2.1	The Box–Muller method	482
C.2.2	The polar method	482
C.3	Notes and further reading	482
	<i>References</i>	483
	<i>Author index</i>	487
	<i>Index</i>	488

Preface

Life insurance has undergone enormous change in the last two to three decades. New and innovative products have been developed at the same time as we have seen vast increases in computational power. In addition, the field of finance has experienced a revolution in the development of a mathematical theory of options and financial guarantees, first pioneered in the work of Black, Scholes and Merton, and actuaries have come to realize the importance of that work to risk management in actuarial contexts.

Given the changes occurring in the interconnected worlds of finance and life insurance, we believe that this is a good time to recast the mathematics of life contingent risk to be better adapted to the products, science and technology that are relevant to current and future actuaries.

In this book we have developed the theory to measure and manage risks that are contingent on demographic experience as well as on financial variables. The material is presented with a certain level of mathematical rigour; we intend for readers to understand the principles involved, rather than to memorize methods or formulae. The reason is that a rigorous approach will prove more useful in the long run than a short-term utilitarian outlook, as theory can be adapted to changing products and technology in ways that techniques, without scientific support, cannot.

We start from a traditional approach, and then develop a more contemporary perspective. The first seven chapters set the context for the material, and cover traditional actuarial models and theory of life contingencies, with modern computational techniques integrated throughout, and with an emphasis on the practical context for the survival models and valuation methods presented. Through the focus on realistic contracts and assumptions, we aim to foster a general business awareness in the life insurance context, at the same time as we develop the mathematical tools for risk management in that context.

In Chapter 8 we introduce multiple state models, which generalize the life–death contingency structure of previous chapters. Using multiple state models allows a single framework for a wide range of insurance, including benefits which depend on health status, on cause of death benefits, or on two or more lives.

In Chapter 9 we apply the theory developed in the earlier chapters to problems involving pension benefits. Pension mathematics has some specialized concepts, particularly in funding principles, but in general this chapter is an application of the theory in the preceding chapters.

In Chapter 10 we move to a more sophisticated view of interest rate models and interest rate risk. In this chapter we explore the crucially important difference between diversifiable and non-diversifiable risk. Investment risk represents a source of non-diversifiable risk, and in this chapter we show how we can reduce the risk by matching cash flows from assets and liabilities.

In Chapter 11 we continue the cash flow approach, developing the emerging cash flows for traditional insurance products. One of the liberating aspects of the computer revolution for actuaries is that we are no longer required to summarize complex benefits in a single actuarial value; we can go much further in projecting the cash flows to see how and when surplus will emerge. This is much richer information that the actuary can use to assess profitability and to better manage portfolio assets and liabilities.

In Chapter 12 we repeat the emerging cash flow approach, but here we look at equity-linked contracts, where a financial guarantee is commonly part of the contingent benefit. The real risks for such products can only be assessed taking the random variation in potential outcomes into consideration, and we demonstrate this with Monte Carlo simulation of the emerging cash flows.

The products that are explored in Chapter 12 contain financial guarantees embedded in the life contingent benefits. Option theory is the mathematics of valuation and risk management of financial guarantees. In Chapter 13 we introduce the fundamental assumptions and results of option theory.

In Chapter 14 we apply option theory to the embedded options of financial guarantees in insurance products. The theory can be used for pricing and for determining appropriate reserves, as well as for assessing profitability.

The material in this book is designed for undergraduate and graduate programmes in actuarial science, and for those self-studying for professional actuarial exams. Students should have sufficient background in probability to be able to calculate moments of functions of one or two random variables, and to handle conditional expectations and variances. We also assume familiarity with the binomial, uniform, exponential, normal and lognormal distributions. Some of the more important results are reviewed in Appendix A. We also assume

that readers have completed an introductory level course in the mathematics of finance, and are aware of the actuarial notation for annuities-certain.

Throughout, we have opted to use examples that liberally call on spreadsheet-style software. Spreadsheets are ubiquitous tools in actuarial practice, and it is natural to use them throughout, allowing us to use more realistic examples, rather than having to simplify for the sake of mathematical tractability. Other software could be used equally effectively, but spreadsheets represent a fairly universal language that is easily accessible. To keep the computation requirements reasonable, we have ensured that every example and exercise can be completed in Microsoft Excel, without needing any VBA code or macros. Readers who have sufficient familiarity to write their own code may find more efficient solutions than those that we have presented, but our principle was that no reader should need to know more than the basic Excel functions and applications. It will be very useful for anyone working through the material of this book to construct their own spreadsheet tables as they work through the first seven chapters, to generate mortality and actuarial functions for a range of mortality models and interest rates. In the worked examples in the text, we have worked with greater accuracy than we record, so there will be some differences from rounding when working with intermediate figures.

One of the advantages of spreadsheets is the ease of implementation of numerical integration algorithms. We assume that students are aware of the principles of numerical integration, and we give some of the most useful algorithms in Appendix B.

The material in this book is appropriate for two one-semester courses. The first seven chapters form a fairly traditional basis, and would reasonably constitute a first course. Chapters 8–14 introduce more contemporary material. Chapter 13 may be omitted by readers who have studied an introductory course covering pricing and delta hedging in a Black–Scholes–Merton model. Chapter 9, on pension mathematics, is not required for subsequent chapters, and could be omitted if a single focus on life insurance is preferred.

Acknowledgements

Many of our students and colleagues have made valuable comments on earlier drafts of parts of the book. Particular thanks go to Carole Bernard, Phelim Boyle, Johnny Li, Ana Maria Mera, Kok Keng Siaw and Matthew Till.

The authors gratefully acknowledge the contribution of the Departments of Statistics and Actuarial Science, University of Waterloo, and Actuarial Mathematics and Statistics, Heriot-Watt University, in welcoming the non-resident

authors for short visits to work on this book. These visits significantly shortened the time it has taken to write the book (to only one year beyond the original deadline).

David Dickson
University of Melbourne

Mary Hardy
University of Waterloo

Howard Waters
Heriot-Watt University

