International Series on Actuarial Science

Actuarial Mathematics for Life Contingent Risks

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Actuarial Mathematics for Life Contingent Risks

How can actuaries best equip themselves for the products and risk structures of the future? In this new textbook, three leaders in actuarial science give a modern perspective on life contingencies.

The book begins traditionally, covering actuarial models and theory, and emphasizing practical applications using computational techniques. The authors then develop a more contemporary outlook, introducing multiple state models, emerging cash flows and embedded options. Using spreadsheet-style software, the book presents large-scale, realistic examples. Over 150 exercises and solutions teach skills in simulation and projection through computational practice.

Balancing rigour with intuition, and emphasizing applications, this textbook is ideal not only for university courses, but also for individuals preparing for professional actuarial examinations and qualified actuaries wishing to renew and update their skills.

International Series on Actuarial Science

The *International Series on Actuarial Science*, published by Cambridge University Press in conjunction with the Institute of Actuaries and the Faculty of Actuaries, contains textbooks for students taking courses in or related to actuarial science, as well as more advanced works designed for continuing professional development or for describing and synthesizing research. The series is a vehicle for publishing books that reflect changes and developments in the curriculum, that encourage the introduction of courses on actuarial science in universities, and that show how actuarial science can be used in all areas where there is long-term financial risk.

ACTUARIAL MATHEMATICS FOR LIFE CONTINGENT RISKS

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To
Carolann,
Vivien
and Phelim

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Preface

Life insurance has undergone enormous change in the last two to three decades. New and innovative products have been developed at the same time as we have seen vast increases in computational power. In addition, the field of finance has experienced a revolution in the development of a mathematical theory of options and financial guarantees, first pioneered in the work of Black, Scholes and Merton, and actuaries have come to realize the importance of that work to risk management in actuarial contexts.

Given the changes occurring in the interconnected worlds of finance and life insurance, we believe that this is a good time to recast the mathematics of life contingent risk to be better adapted to the products, science and technology that are relevant to current and future actuaries.

In this book we have developed the theory to measure and manage risks that are contingent on demographic experience as well as on financial variables. The material is presented with a certain level of mathematical rigour; we intend for readers to understand the principles involved, rather than to memorize methods or formulae. The reason is that a rigorous approach will prove more useful in the long run than a short-term utilitarian outlook, as theory can be adapted to changing products and technology in ways that techniques, without scientific support, cannot.

We start from a traditional approach, and then develop a more contemporary perspective. The first seven chapters set the context for the material, and cover traditional actuarial models and theory of life contingencies, with modern computational techniques integrated throughout, and with an emphasis on the practical context for the survival models and valuation methods presented. Through the focus on realistic contracts and assumptions, we aim to foster a general business awareness in the life insurance context, at the same time as we develop the mathematical tools for risk management in that context.

In Chapter 8 we introduce multiple state models, which generalize the life–death contingency structure of previous chapters. Using multiple state models allows a single framework for a wide range of insurance, including benefits which depend on health status, on cause of death benefits, or on two or more lives.

In Chapter 9 we apply the theory developed in the earlier chapters to problems involving pension benefits. Pension mathematics has some specialized concepts, particularly in funding principles, but in general this chapter is an application of the theory in the preceding chapters.

In Chapter 10 we move to a more sophisticated view of interest rate models and interest rate risk. In this chapter we explore the crucially important difference between diversifiable and non-diversifiable risk. Investment risk represents a source of non-diversifiable risk, and in this chapter we show how we can reduce the risk by matching cash flows from assets and liabilities.

In Chapter 11 we continue the cash flow approach, developing the emerging cash flows for traditional insurance products. One of the liberating aspects of the computer revolution for actuaries is that we are no longer required to summarize complex benefits in a single actuarial value; we can go much further in projecting the cash flows to see how and when surplus will emerge. This is much richer information that the actuary can use to assess profitability and to better manage portfolio assets and liabilities.

In Chapter 12 we repeat the emerging cash flow approach, but here we look at equity-linked contracts, where a financial guarantee is commonly part of the contingent benefit. The real risks for such products can only be assessed taking the random variation in potential outcomes into consideration, and we demonstrate this with Monte Carlo simulation of the emerging cash flows.

The products that are explored in Chapter 12 contain financial guarantees embedded in the life contingent benefits. Option theory is the mathematics of valuation and risk management of financial guarantees. In Chapter 13 we introduce the fundamental assumptions and results of option theory.

In Chapter 14 we apply option theory to the embedded options of financial guarantees in insurance products. The theory can be used for pricing and for determining appropriate reserves, as well as for assessing profitability.

The material in this book is designed for undergraduate and graduate programmes in actuarial science, and for those self-studying for professional actuarial exams. Students should have sufficient background in probability to be able to calculate moments of functions of one or two random variables, and to handle conditional expectations and variances. We also assume familiarity with the binomial, uniform, exponential, normal and lognormal distributions. Some of the more important results are reviewed in Appendix A. We also assume that readers have completed an introductory level course in the mathematics of finance, and are aware of the actuarial notation for annuities-certain.

Throughout, we have opted to use examples that liberally call on spreadsheet-style software. Spreadsheets are ubiquitous tools in actuarial practice, and it is natural to use them throughout, allowing us to use more realistic examples, rather than having to simplify for the sake of mathematical tractability. Other software could be used equally effectively, but spreadsheets represent a fairly universal language that is easily accessible. To keep the computation requirements reasonable, we have ensured that every example and exercise can be completed in Microsoft Excel, without needing any VBA code or macros. Readers who have sufficient familiarity to write their own code may find more efficient solutions than those that we have presented, but our principle was that no reader should need to know more than the basic Excel functions and applications. It will be very useful for anyone working through the material of this book to construct their own spreadsheet tables as they work through the first seven chapters, to generate mortality and actuarial functions for a range of mortality models and interest rates. In the worked examples in the text, we have worked with greater accuracy than we record, so there will be some differences from rounding when working with intermediate figures.

One of the advantages of spreadsheets is the ease of implementation of numerical integration algorithms. We assume that students are aware of the principles of numerical integration, and we give some of the most useful algorithms in Appendix B.

The material in this book is appropriate for two one-semester courses. The first seven chapters form a fairly traditional basis, and would reasonably constitute a first course. Chapters 8–14 introduce more contemporary material. Chapter 13 may be omitted by readers who have studied an introductory course covering pricing and delta hedging in a Black–Scholes–Merton model. Chapter 9, on pension mathematics, is not required for subsequent chapters, and could be omitted if a single focus on life insurance is preferred.

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