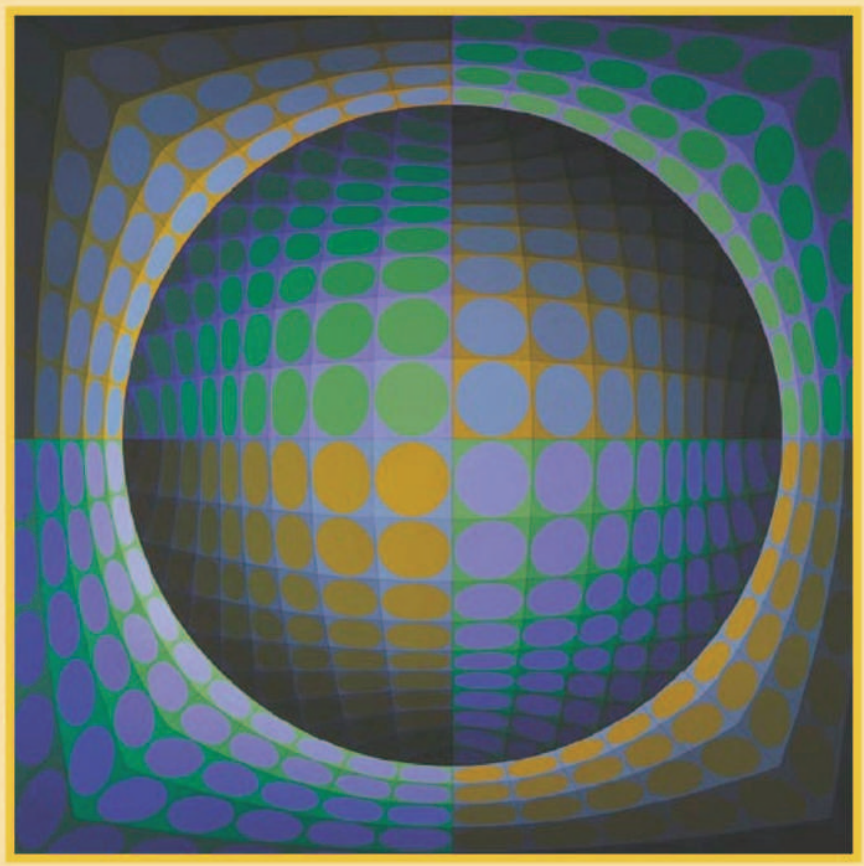
ALGORITHMS of Informatics



volume 2

ALGORITHMS

OF INFORMATICS

Volume 2

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Editor: Antal Iványi

Authors of Volume 1: László Lovász (Preface), Antal Iványi (Introduction), Zoltán Kása (Chapter 1), Zoltán Csörnyei (Chapter 2), Ulrich Tamm (Chapter 3), Péter Gács (Chapter 4), Gábor Ivanyos and Lajos Rónyai (Chapter 5), Antal Járai and Attila Kovács (Chapter 6), Jörg Rothe (Chapters 7 and 8), Csanád Imreh (Chapter 9), Ferenc Szidarovszky (Chapter 10), Zoltán Kása (Chapter 11), Aurél Galántai and András Jeney (Chapter 12),

Validators of Volume 1: Zoltán Fülöp (Chapter 1), Pál Dömösi (Chapter 2), Sándor Fridli (Chapter 3), Anna Gál (Chapter 4), Attila Pethő (Chapter 5), Lajos Rónyai (Chapter 6), János Gonda (Chapter 7), Gábor Ivanyos (Chapter 8), Béla Vizvári (Chapter 9), János Mayer (Chapter 10), András Recski (Chapter 11), Tamás Szántai (Chapter 12), Anna Iványi (Bibliography)

Authors of Volume 2: Burkhard Englert, Dariusz Kowalski, Gregorz Malewicz, and Alexander Shvartsman (Chapter 13), Tibor Gyires (Chapter 14), Claudia Fohry and Antal Iványi (Chapter 15), Eberhard Zehendner (Chapter 16), Ádám Balogh and Antal Iványi (Chapter 17), János Demetrovics and Attila Sali (Chapters 18 and 19), Attila Kiss (Chapter 20), István Miklós (Chapter 21), László Szirmay-Kalos (Chapter 22), Ingo Althöfer and Stefan Schwarz (Chapter 23)

Validators of Volume 2: István Majzik (Chapter 13), János Sztrik (Chapter 14), Dezső Sima (Chapters 15 and 16), László Varga (Chapter 17), Attila Kiss (Chapters 18 and 19), András Benczúr (Chapter 20), István Katsányi (Chapter 21), János Vida (Chapter 22), Tamás Szántai (Chapter 23), Anna Iványi (Bibliography)

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IV. COMPUTER NETWORKS

13. Distributed Algorithms

We define a distributed system as a collection of individual computing devices that can communicate with each other. This definition is very broad, it includes anything, from a VLSI chip, to a tightly coupled multiprocessor, to a local area cluster of workstations, to the Internet. Here we focus on more loosely coupled systems. In a distributed system as we view it, each processor has its semi-independent agenda, but for various reasons, such as sharing of resources, availability, and fault-tolerance, processors need to coordinate their actions.
Distributed systems are highly desirable, but it is notoriously difficult to construct efficient distributed algorithms that perform well in realistic system settings. These difficulties are not just of a more practical nature, they are also fundamental in nature. In particular, many of the difficulties are introduced by the three factors of: asynchrony, limited local knowledge, and failures. Asynchrony means that global time may not be available, and that both absolute and relative times at which events take place at individual computing devices can often not be known precisely. Moreover, each computing device can only be aware of the information it receives, it has therefore an inherently local view of the global status of the system. Finally, computing devices and network components may fail independently, so that some remain functional while others do not.

We will begin by describing the models used to analyse distributed systems in the message-passing model of computation. We present and analyze selected distributed algorithms based on these models. We include a discussion of fault-tolerance in distributed systems and consider several algorithms for reaching agreement in the messages-passing models for settings prone to failures. Given that global time is often unavailable in distributed systems, we present approaches for providing logical time that allows one to reason about causality and consistent states in distributed systems. Moving on to more advanced topics, we present a spectrum of broadcast services often considered in distributed systems and present algorithms implementing these services. We also present advanced algorithms for rumor gathering algorithms. Finally, we also consider the mutual exclusion problem in the shared-memory model of distributed computation.

13.1. Message passing systems and algorithms

We present our first model of distributed computation, for message passing systems without failures. We consider both synchronous and asynchronous systems and present selected algorithms for message passing systems with arbitrary network topology, and both synchronous and asynchronous settings.

13.1.1. Modeling message passing systems

In a message passing system, processors communicate by sending messages over communication channels, where each channel provides a bidirectional connection between two specific processors. We call the pattern of connections described by the channels, the ***topology*** of the system. This topology is represented by an undirected graph, where each node represents a processor, and an edge is present between two nodes if and only if there is a channel between the two processors represented by the nodes. The collection of channels is also called the ***network.*** An algorithm for such a message passing system with a specific topology consists of a local program for each processor in the system. This local program provides the ability to the processor to perform local computations, to send and receive messages from each of its neighbours in the given topology.

Each processor in the system is modeled as a possibly infinite state machine. A ***configuration*** is a vector $C = (q_0, \ldots, q_{n-1})$ where each q_i is the state of a processor p_i. Activities that can take place in the system are modeled as ***events*** (or *actions*) that describe indivisible system operations. Examples of events include local computation events and delivery events where a processor receives a message. The behaviour of the system over time is modeled as an ***execution,*** a (finite or infinite) sequence of configurations (C_i) alternating with events (a_i): $C_0, a_1, C_1, a_2, C_2, \ldots$. Executions must satisfy a variety of conditions that are used to represent the correctness properties, depending on the system being modeled. These conditions can be classified as either safety or liveness conditions. A ***safety condition*** for a system is a condition that must hold in every finite prefix of any execution of the system. Informally it states that nothing *bad* has happened yet. A ***liveness condition*** is a condition that must hold a certain (possibly infinite) number of times. Informally it states that eventually something *good* must happen. An important liveness condition is ***fairness,*** which requires that an (infinite) execution contains infinitely many actions by a processor, unless after some configuration no actions are enabled at that processor.

13.1.2. Asynchronous systems

We say that a system is ***asynchronous*** if there is no fixed upper bound on how long it takes for a message to be delivered or how much time elapses between consecutive steps of a processor. An obvious example of such an asynchronous system is the Internet. In an implementation of a distributed system there are often upper bounds on message delays and processor step times. But since these upper bounds are often very large and can change over time, it is often desirable to develop an algorithm

that is independent of any timing parameters, that is, an asynchronous algorithm.

In the asynchronous model we say that an execution is ***admissible*** if each processor has an infinite number of computation events, and every message sent is eventually delivered. The first of these requirements models the fact that processors do not fail. (It does not mean that a processor's local program contains an infinite loop. An algorithm can still terminate by having a transition function not change a processors state after a certain point.)

We assume that each processor's set of states includes a subset of ***terminated*** states. Once a processor enters such a state it remains in it. The algorithm has ***terminated*** if all processors are in terminated states and no messages are in transit.

The ***message complexity*** of an algorithm in the asynchronous model is the maximum over all admissible executions of the algorithm, of the total number of (point-to-point) messages sent.

A ***timed execution*** is an execution that has a nonnegative real number associated with each event, the *time* at which the event occurs. To measure the *time complexity* of an asynchronous algorithm we first assume that the maximum message delay in any execution is one unit of time. Hence the ***time complexity*** is the maximum time until termination among all timed admissible executions in which every message delay is at most one. Intuitively this can be viewed as taking any execution of the algorithm and normalising it in such a way that the longest message delay becomes one unit of time.

13.1.3. Synchronous systems

In the synchronous model processors execute in lock-step. The execution is partitioned into rounds so that every processor can send a message to each neighbour, the messages are delivered, and every processor computes based on the messages just received. This model is very convenient for designing algorithms. Algorithms designed in this model can in many cases be automatically simulated to work in other, more realistic timing models.

In the synchronous model we say that an execution is admissible if it is infinite. From the round structure it follows then that every processor takes an infinite number of computation steps and that every message sent is eventually delivered. Hence in a synchronous system with no failures, once a (deterministic) algorithm has been fixed, the only relevant aspect determining an execution that can change is the initial configuration. On the other hand in an asynchronous system, there can be many different executions of the same algorithm, even with the same initial configuration and no failures, since here the interleaving of processor steps, and the message delays, are not fixed.

The notion of *terminated states* and the *termination* of the algorithm is defined in the same way as in the asynchronous model.

The *message complexity* of an algorithm in the synchronous model is the maximum over all admissible executions of the algorithm, of the total number of messages sent.

To measure time in a synchronous system we simply count the number of rounds until termination. Hence the ***time complexity*** of an algorithm in the synchronous

model is the maximum number of rounds in any admissible execution of the algorithm until the algorithm has terminated.

13.2. Basic algorithms

We begin with some simple examples of algorithms in the message passing model.

13.2.1. Broadcast

We start with a simple algorithm Spanning-Tree-Broadcast for the (single message) broadcast problem, assuming that a spanning tree of the network graph with n nodes (processors) is already given. Later, we will remove this assumption. A processor p_i wishes to send a message M to all other processors. The spanning tree rooted at p_i is maintained in a distributed fashion: Each processor has a distinguished channel that leads to its *parent* in the tree as well as a set of channels that lead to its *children* in the tree. The root p_i sends the message M on all channels leading to its children. When a processor receives the message on a channel from its parent, it sends M on all channels leading to its children.

Spanning-Tree-Broadcast

```
   Initially M is in transit from p_i to all its children in the spanning tree.
   Code for p_i:
1      upon receiving no message: // first computation event by p_i
2          terminate

   Code for p_j, 0 ≤ j ≤ n − 1, j ≠ i:
3      upon receiving M from parent:
4          send M to all children
5          terminate
```

The algorithm Spanning-Tree-Broadcast is correct whether the system is synchronous or asynchronous. Moreover, the message and time complexities are the same in both models.

Using simple inductive arguments we will first prove a lemma that shows that by the end of round t, the message M reaches all processors at distance t (or less) from p_r in the spanning tree.

Lemma 13.1 *In every admissible execution of the broadcast algorithm in the synchronous model, every processor at distance t from p_r in the spanning tree receives the message M in round t.*

Proof We proceed by induction on the distance t of a processor from p_r. First let $t = 1$. It follows from the algorithm that each child of p_r receives the message in round 1.

Assume that each processor at distance $t - 1$ received the message M in round

$t-1$. We need to show that each processor p_t at distance t receives the message in round t. Let p_s be the parent of p_t in the spanning tree. Since p_s is at distance $t-1$ from p_r, by the induction hypothesis, p_s received M in round $t-1$. By the algorithm, p_t will hence receive M in round t. ∎

By Lemma 13.1 the time complexity of the broadcast algorithm is d, where d is the depth of the spanning tree. Now since d is at most $n-1$ (when the spanning tree is a chain) we have:

Theorem 13.2 *There is a synchronous broadcast algorithm for n processors with message complexity $n-1$ and time complexity d, when a rooted spanning tree with depth d is known in advance.*

We now move to an asynchronous system and apply a similar analysis.

Lemma 13.3 *In every admissible execution of the broadcast algorithm in the asynchronous model, every processor at distance t from p_r in the spanning tree receives the message M by time t.*

Proof We proceed by induction on the distance t of a processor from p_r. First let $t=1$. It follows from the algorithm that M is initially in transit to each processor p_i at distance 1 from p_r. By the definition of time complexity for the asynchronous model, p_i receives M by time 1.

Assume that each processor at distance $t-1$ received the message M at time $t-1$. We need to show that each processor p_t at distance t receives the message by time t. Let p_s be the parent of p_t in the spanning tree. Since p_s is at distance $t-1$ from p_r, by the induction hypothesis, p_s sends M to p_t when it receives M at time $t-1$. By the algorithm, p_t will hence receive M by time t. ∎

We immediately obtain:

Theorem 13.4 *There is an asynchronous broadcast algorithm for n processors with message complexity $n-1$ and time complexity d, when a rooted spanning tree with depth d is known in advance.*

13.2.2. Construction of a spanning tree

The asynchronous algorithm called Flood, discussed next, constructs a spanning tree rooted at a designated processor p_r. The algorithm is similar to the Depth First Search (DFS) algorithm. However, unlike DFS where there is just one processor with "global knowledge" about the graph, in the Flood algorithm, each processor has "local knowledge" about the graph, processors coordinate their work by exchanging messages, and processors and messages may get delayed arbitrarily. This makes the design and analysis of Flood algorithm challenging, because we need to show that the algorithm indeed constructs a spanning tree despite conspiratorial selection of these delays.

Algorithm description. Each processor has four local variables. The links adjacent to a processor are identified with distinct numbers starting from 1 and stored in a local variable called *neighbours*. We will say that the ***spanning tree has been constructed.*** when the variable *parent* stores the identifier of the link leading to the parent of the processor in the spanning tree, except that this variable is NONE for the designated processor p_r; *children* is a set of identifiers of the links leading to the children processors in the tree; and *other* is a set of identifiers of all other links. So the knowledge about the spanning tree may be "distributed" across processors.

The code of each processor is composed of segments. There is a segment (lines 1–4) that describes how local variables of a processor are initialised. Recall that the local variables are initialised that way before time 0. The next three segments (lines 5–11, 12–15 and 16–19) describe the instructions that any processor executes in response to having received a message: <*adopt*>, <*approved*> or <*rejected*>. The last segment (lines 20–22) is only included in the code of processor p_r. This segment is executed only when the local variable *parent* of processor p_r is NIL. At some point of time, it may happen that more than one segment can be executed by a processor (e.g., because the processor received <*adopt*> messages from two processors). Then the processor executes the segments serially, one by one (segments of any given processor are never executed concurrently). However, instructions of different processor may be arbitrarily interleaved during an execution. Every message that can be processed is eventually processed and every segment that can be executed is eventually executed (fairness).

FLOOD

```
     Code for any processor p_k, 1 ≤ k ≤ n
 1   initialisation
 2       parent ← NIL
 3       children ← ∅
 4       other ← ∅

 5   process message <adopt> that has arrived on link j
 6       if parent = NIL
 7         then parent ← j
 8              send <approved> to link j
 9              send <adopt> to all links in neighbours \ {j}
10         else send <rejected> to link j

11   process message <approved> that has arrived on link j
12       children ← children ∪ {j}
13       if children ∪ other = neighbours \ {parent}
14         then terminate
```

```
15  process message <rejected> that has arrived on link j
16      other ← other ∪ {j}
17      if children ∪ other = neighbours \ {parent}
18          then terminate
    Extra code for the designated processor p_r
19  if parent = NIL
20      then parent ← NONE
21           send <adopt> to all links in neighbours
```

Let us outline how the algorithm works. The designated processor sends an $<$*adopt*$>$ message to all its neighbours, and assigns NONE to the *parent* variable (NIL and NONE are two distinguished values, different from any natural number), so that it never again sends the message to any neighbour.

When a processor processes message $<$*adopt*$>$ for the first time, the processor assigns to its own *parent* variable the identifier of the link on which the message has arrived, responds with an $<$*approved*$>$ message to that link, and forwards an $<$*adopt*$>$ message to every other link. However, when a processor processes message $<$*adopt*$>$ again, then the processor responds with a $<$*rejected*$>$ message, because the *parent* variable is no longer NIL.

When a processor processes message $<$*approved*$>$, it adds the identifier of the link on which the message has arrived to the set *children*. It may turn out that the sets *children* and *other* combined form identifiers of all links adjacent to the processor except for the identifier stored in the *parent* variable. In this case the processor enters a terminating state.

When a processor processes message $<$*rejected*$>$, the identifier of the link is added to the set *other*. Again, when the union of *children* and *other* is large enough, the processor enters a terminating state.

Correctness proof. We now argue that FLOOD constructs a spanning tree. The key moments in the execution of the algorithm are when any processor assigns a value to its *parent* variable. These assignments determine the "shape" of the spanning tree. The facts that any processor eventually executes an instruction, any message is eventually delivered, and any message is eventually processed, ensure that the knowledge about these assignments spreads to neighbours. Thus the algorithm is expanding a subtree of the graph, albeit the expansion may be slow. Eventually, a spanning tree is formed. Once a spanning tree has been constructed, eventually every processor will terminate, even though some processors may have terminated even before the spanning tree has been constructed.

Lemma 13.5 *For any $1 \leq k \leq n$, there is time t_k which is the first moment when there are exactly k processors whose parent variables are not* NIL*, and these processors and their parent variables form a tree rooted at p_r.*

Proof We prove the statement of the lemma by induction on k. For the base case, assume that $k = 1$. Observe that processor p_r eventually assigns NONE to its *parent*

variable. Let t_1 be the moment when this assignment happens. At that time, the *parent* variable of any processor other than p_r is still NIL, because no <*adopt*> messages have been sent so far. Processor p_r and its *parent* variable form a tree with a single node and not arcs. Hence they form a rooted tree. Thus the inductive hypothesis holds for $k = 1$.

For the inductive step, suppose that $1 \leq k < n$ and that the inductive hypothesis holds for k. Consider the time t_k which is the first moment when there are exactly k processors whose *parent* variables are not *nil*. Because $k < n$, there is a non-tree processor. But the graph G is connected, so there is a non-tree processor adjacent to the tree. (For any subset T of processors, a processor p_i is ***adjacent*** to T if and only if there an edge in the graph G from p_i to a processor in T.) Recall that by definition, *parent* variable of such processor is NIL. By the inductive hypothesis, the k processors must have executed line 7 of their code, and so each either has already sent or will eventually send <*adopt*> message to all its neighbours on links other than the *parent* link. So the non-tree processors adjacent to the tree have already received or will eventually receive <*adopt*> messages. Eventually, each of these adjacent processors will, therefore, assign a value other than NIL to its *parent* variable. Let $t_{k+1} > t_k$ be the first moment when any processor performs such assignment, and let us denote this processor by p_i. This cannot be a tree processor, because such processor never again assigns any value to its *parent* variable. Could p_i be a non-tree processor that is not adjacent to the tree? It could not, because such processor does not have a direct link to a tree processor, so it cannot receive <*adopt*> directly from the tree, and so this would mean that at some time t' between t_k and t_{k+1} some other non-tree processor p_j must have sent <*adopt*> message to p_i, and so p_j would have to assign a value other than NIL to its *parent* variable some time after t_k but before t_{k+1}, contradicting the fact the t_{k+1} is the first such moment. Consequently, p_i is a non-tree processor adjacent to the tree, such that, at time t_{k+1}, p_i assigns to its *parent* variable the index of a link leading to a tree processor. Therefore, time t_{k+1} is the first moment when there are exactly $k + 1$ processors whose *parent* variables are not NIL, and, at that time, these processors and their *parent* variables form a tree rooted at p_r. This completes the inductive step, and the proof of the lemma. ■

Theorem 13.6 *Eventually each processor terminates, and when every processor has terminated, the subgraph induced by the parent variables forms a spanning tree rooted at p_r.*

Proof By Lemma 13.5, we know that there is a moment t_n which is the first moment when all processors and their *parent* variables form a spanning tree.

Is it possible that every processor has terminated before time t_n? By inspecting the code, we see that a processor terminates only after it has received <*rejected*> or <*approved*> messages from all its neighbours other than the one to which *parent* link leads. A processor receives such messages only in response to <*adopt*> messages that the processor sends. At time t_n, there is a processor that still has not even sent <*adopt*> messages. Hence, not every processor has terminated by time t_n.

Will every processor eventually terminate? We notice that by time t_n, each processor either has already sent or will eventually send <*adopt*> message to all

its neighbours other than the one to which *parent* link leads. Whenever a processor receives <*adopt*> message, the processor responds with <*rejected*> or <*approved*>, even if the processor has already terminated. Hence, eventually, each processor will receive either <*rejected*> or <*approved*> message on each link to which the processor has sent <*adopt*> message. Thus, eventually, each processor terminates. ■

We note that the fact that a processor has terminated does not mean that a spanning tree has already been constructed. In fact, it may happen that processors in a different part of the network have not even received any message, let alone terminated.

Theorem 13.7 *Message complexity of* FLOOD *is* $O(e)$*, where* e *is the number of edges in the graph* G*.*

The proof of this theorem is left as Problem 13-1.

Exercises

13.2-1 It may happen that a processor has terminated even though a processor has not even received any message. Show a simple network and how to delay message delivery and processor computation to demonstrate that this can indeed happen.

13.2-2 It may happen that a processor has terminated but may still respond to a message. Show a simple network and how to delay message delivery and processor computation to demonstrate that this can indeed happen.

13.3. Ring algorithms

One often needs to coordinate the activities of processors in a distributed system. This can frequently be simplified when there is a single processor that acts as a coordinator. Initially, the system may not have any coordinator, or an existing coordinator may fail and so another may need to be elected. This creates the problem where processors must elect exactly one among them, a *leader*. In this section we study the problem for special types of networks—rings. We will develop an asynchronous algorithm for the problem. As we shall demonstrate, the algorithm has asymptotically optimal message complexity. In the current section, we will see a distributed analogue of the well-known divide-and-conquer technique often used in sequential algorithms to keep their time complexity low. The technique used in distributed systems helps reduce the message complexity.

13.3.1. The leader election problem

The leader election problem is to elect exactly leader among a set of processors. Formally each processor has a local variable *leader* initially equal to NIL. An algorithm is said to ***solve the leader election problem*** if it satisfies the following conditions:

1. in any execution, exactly one processor eventually assigns TRUE to its *leader* variable, all other processors eventually assign FALSE to their *leader* variables,